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Restoration of chiral symmetry: a supergravity perspective

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Abstract: The supergravity dual of N regular and M fractional D3-branes on the conifold has a naked singularity in the infrared. Supersymmetric resolution of this singularity requires deforming the conifold: this is the supergravity dual of chiral symmetry breaking. Buchel suggested that at sufficiently high temperature there is no need to deform the conifold: the singularity may be cloaked by a horizon. This would be the supergravity manifestation of chiral symmetry restoration. In previous work [hep-th/0102105] the ansatz and the system of second-order radial differential equations necessary to find such a solution were written down. In this paper we find smooth solutions to this system in a perturbation theory that is valid when the Hawking temperature of the horizon is very high.

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Contents

1. Introduction

The AdS/CFT correspondence [1, 2, 3] has produced a wealth of new information about strongly coupled conformal gauge theories. Considerable effort has also been invested into extending it to non-conformal theories. One recent development is the supergravity description of chiral symmetry breaking in a certain $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)\times\mathrm{SU}(N+M)$ gauge theory [4]. This theory may be realized by adding M fractional D3-branes (wrapped D5-branes) to N regular D3-branes at the apex of the conifold, which is defined by the constraint $\sum_{i=1}^{4} z_i^2 = 0$ in $\mathbb{C}^4$. For $M=0$ this gauge theory reduces to the superconformal theory dual to the $AdS_5\times\mathbb{T}^{1,1}$ background of type-IIB string [5, 6].

In the supergravity dual the M fractional branes are replaced by M units of RR 3-form flux through the 3-cycle of the compact space. This flux changes the background and introduces the logarithmic running of $\int_{S^2} B_2$, which is related to the running of field theoretic couplings [7]. In turn, this causes the RR 5-form flux to grow logarithmically with the radius [8], due to the equation $dF_5 = H_3\wedge F_3$. In [4] this behavior was attributed to a cascade of Seiberg dualities in the dual $\mathcal{N}=1$ supersymmetric $\mathrm{SU}(N)\times\mathrm{SU}(N+M)$ gauge theory.

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While the solution of [8] is smooth in the UV (for large ρ), it has a naked singularity in the IR. To resolve this singularity while preserving the $\mathcal{N}=1$ supersymmetry, it is necessary to deform the conifold [4], i.e. to replace the constraint with $\sum_{i=1}^{4} z_i^2 = \epsilon^2$. The resulting solution, a warped deformed conifold, is perfectly non-singular and without a horizon in the IR, while it asymptotically approaches the KT solution [8] in the UV. The mechanism that removes the naked singularity is related to the breaking of the chiral symmetry in the dual $\mathrm{SU}(N) \times \mathrm{SU}(N+M)$ gauge theory. The $\mathbb{Z}_{2M}$ chiral symmetry, which may be approximated by U(1) for large M, is realized geometrically as $z_i \to z_i e^{i\theta}$. The deformation of the conifold breaks it down to $\mathbb{Z}_2$: $z_i \to -z_i$ [4].

In [9] a different mechanism for resolving the KT naked singularity was proposed. It was suggested that a non-extremal generalization of the KT solution, with unbroken U(1) symmetry, may have a regular Schwarzschild horizon "cloaking" the naked singularity. The dual field theory interpretation of this would be the restoration of chiral symmetry above some critical temperature T_c [9]. The proposal of [9] is that the description of the phase with restored symmetry involves a regular Schwarzschild horizon appearing in the asymptotically KT geometry. This proposal is analogous to the fact that the $\mathcal{N}=4$ SYM theory, which is not confining, is described at a finite temperature by a black hole in AdS_5 [10, 11].

We may ask, to what extent can chiral symmetry breaking and confinement be studied, for fractional D3-branes on the conifold, via black hole horizons? It is the aim of this paper to demonstrate that at least the high temperature phase is accessible (thus realizing the goal of [9]). The thermodynamics of the low temperature phase might be difficult to treat from a classical supergravity perspective. The reason is that at low temperatures, where confinement has occurred, there are $O(N^0)$ degrees of freedom, whereas on the classical supergravity side, a horizon has an entropy of $O(N^2)$, where N is the relevant number of colors. So it would seem that to study the thermodynamics of the $T < T_\mathrm{c}$ phase we need to go at least to one-loop effects on the string theory side.

We leave open the question of how well one might describe in supergravity terms the phase transition where chiral symmetry is broken. Standard lore suggests that this is a second order transition at finite T_c. The geometry dual to the theory at criticality should admit some action of the conformal group; or at the least, some correlation functions, computed with periodic euclidean time, should have power law tails in the infrared. Now, the latter statement is impossible as long as there is a smooth horizon, because in the periodic euclidean time formalism, a smooth horizon means that the non-compact space has the topology of a disk (in the t-r directions) times $\mathbb{R}^3$, and the disk factor will invariably generate a gap, as observed in [11]. The alternative is to have the topology of a cylinder times $\mathbb{R}^3$. This would be the case, for instance, if the geometry dual to the critical point were euclidean AdS_5 with time made periodic. Such a geometry is the obvious candidate to describe criticality,[1] since it does admit an obvious action of the spatial part of the conformal group, which can be preserved at finite temperature. The horizon is degenerate, and by the usual rules of black hole thermodynamics has no entropy — which is only to say that the entropy is subleading in N. The same statements seem to apply whenever the topology in the t-r direction is a cylinder.

The considerations of the previous paragraphs all tend toward the conclusion that a regular black hole horizon should appear only at some finite Hawking temperature, and that this temperature should be the $T_{\rm c}$ of chiral symmetry breaking. For temperatures below $T_{\rm c}$, we might expect non-extremal generalizations of the KS solution which are free of horizons, just like the extremal solution. The absence of a horizon in the extremal geometry results in an area law for Wilson loops, which is a manifestation of confinement. Near-extremal generalizations of it should retain this area-law property. Clearly it would be very interesting to study the $T = T_{\rm c}$ point in supergravity, to the extent that this is possible. We suggest a limiting procedure which may be used for finding the appropriate supergravity background.

In order to address physically interesting questions at finite temperature from a dual supergravity point of view, we need to study non-extremal generalizations of the KT and KS backgrounds. This problem was first addressed for the KT background by Buchel in [9] partly with numerical methods. However, in [12] this solution was shown to be singular. It was argued there that a more general ansatz where the 3-forms are not self-dual is necessary to find a regular Schwarzschild horizon.

In this paper we show that solutions of this type indeed exist. First we review the ansatz and the basic equations derived in [12]. Then we proceed to develop perturbation theory that is valid in the high temperature phase $T \gg T_{\rm c}$.

2. Non-extremal generalization of the KT ansatz

We start with the ansatz of [12] for the non-extremal KT background describing the high temperature phase where the chiral symmetry is restored. The 10-d Einstein-frame metric was taken to be of the general form consistent with the U(1) symmetry of ψ-rotations and the interchange of the two S^2's. It involves 4 functions x, y, z, w of a radial coordinate u:

$$ds^2_{10E} = e^{2z}\left(e^{-6x}dX_0^2 + e^{2x}dX_idX_i\right) + e^{-2z}ds_6^2\,, \tag{2.1}$$

where

$$\begin{aligned} ds_6^2 &= e^{10y}du^2 + e^{2y}(dM_5)^2\,, \\ (dM_5)^2 &= e^{-8w}e_\psi^2 + e^{2w}\left(e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2\right) \equiv e^{2w}ds_5^2\,, \end{aligned} \tag{2.2}$$

[1]It seems difficult however to find an anti-de Sitter solution of supergravity with the fluxes appropriate to fractional D3-branes.

and

$$e_\psi = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)\,, \qquad e_{\theta_i} = \frac{1}{\sqrt{6}} d\theta_i\,, \qquad e_{\phi_i} = \frac{1}{\sqrt{6}} \sin\theta_i d\phi_i\,.$$

Here X_0 is the euclidean time and X_i are the 3 longitudinal 3-brane directions.

This metric can be brought into a more familiar D3-brane form

$$ds^2_{10E} = h^{-1/2}(\rho)\left[g(\rho)dX_0^2 + dX_i dX_i\right] + h^{1/2}(\rho)\left[\mathrm{g}^{-1}(\rho)d\rho^2 + \rho^2 ds_5^2\right]\,, \tag{2.3}$$

with the redefinitions

$$h = e^{-4z-4x}\,, \qquad \rho = e^{y+x+w}\,, \qquad g = e^{-8x}\,, \qquad e^{10y+2x}du^2 = \mathrm{g}^{-1}(\rho)d\rho^2\,. \tag{2.4}$$

When $w = 0$ and $e^{4y} = \rho^4 = 1/4u$, the transverse 6-d space is the standard conifold with $M_5 = \mathbb{T}^{1,1}$. Small u thus corresponds to large distances (where we shall assume that g, g and also h, in the asymptotically flat case, approach 1 as $\rho \to \infty$) and vice versa. The function w squashes the U(1) fiber of $\mathbb{T}^{1,1}$ relative to the 2-spheres; it does not violate the U(1) symmetry.

The ansatz for the p-form fields is dictated by symmetries and thus is exactly the same as in the extremal KT case [8]:[2]

$$\begin{aligned} F_3 &= P e_\psi \wedge (e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2})\,, \\ B_2 &= f(u)(e_{\theta_1} \wedge e_{\phi_1} - e_{\theta_2} \wedge e_{\phi_2})\,, \\ F_5 &= \mathcal{F} + *\mathcal{F}\,, \qquad \mathcal{F} = K(u) e_\psi \wedge e_{\theta_1} \wedge e_{\phi_1} \wedge e_{\theta_2} \wedge e_{\phi_2}\,. \end{aligned} \tag{2.5}$$

As in [8], the Bianchi identity for the 5-form, $d * F_5 = dF_5 = H_3 \wedge F_3$, implies

$$K(u) = Q + 2Pf(u)\,. \tag{2.6}$$

We will not generally impose the self-duality of the 3-forms, which means that we have to include the dynamics of the dilaton Φ and of the "squash factor" w.

3. Basic system of equations and its limiting solutions

The system of radial equations that need to be solved was derived in [12]. The simplest equation is for the "non-extremality function" x

$$x'' = 0\,, \qquad \text{i.e.} \qquad x = au\,, \qquad a = \text{const}\,, \tag{3.1}$$

[2]Note that the function T in [8] is related to f used in [12] and here by $f = \frac{1}{\sqrt{2}}T$, and we make a similar rescaling of P: $P = \frac{1}{\sqrt{2}}P_{\mathrm{KT}}$.

and in the non-extremal case we take $a > 0$. The coupled equations for the remaining five functions of u, i.e. y, z, w, Φ and f or, equivalently, K, are

$$10y'' - 8e^{8y}\left(6e^{-2w} - e^{-12w}\right) + \Phi'' = 0\,, \tag{3.2}$$

$$10w'' - 12e^{8y}\left(e^{-2w} - e^{-12w}\right) - \Phi'' = 0\,, \tag{3.3}$$

$$\Phi'' + e^{-\Phi+4z-4y-4w}\left(\frac{K'^2}{4P^2} - e^{2\Phi+8y+8w}P^2\right) = 0\,, \tag{3.4}$$

$$4z'' - K^2e^{8z} - e^{-\Phi+4z-4y-4w}\left(\frac{K'^2}{4P^2} + e^{2\Phi+8y+8w}P^2\right) = 0\,, \tag{3.5}$$

$$\left(e^{-\Phi+4z-4y-4w}K'\right)' - 2P^2Ke^{8z} = 0\,. \tag{3.6}$$

The integration constants are subject to the zero-energy constraint $T + V = 0$, i.e.[3]

$$\begin{aligned} 5y'^2 - 2z'^2 - 5w'^2 - \frac{1}{8}\Phi'^2 - e^{-\Phi+4z-4y-4w}\frac{K'^2}{16P^2} - \\ - e^{8y}\left(6e^{-2w} - e^{-12w}\right) + \frac{1}{4}e^{\Phi+4z+4y+4w}P^2 + \frac{1}{8}e^{8z}K^2 = 3a^2\,. \end{aligned} \tag{3.7}$$

A note about the dimensions. From the form of the metric (2.1) it is natural to require that e^y and $u^{-1/4}$ should have dimension of length, while x, z, w should be dimensionless. Since we have set the 10-d gravitational constant to be 1 (i.e. we measure the scales in terms of the 10-d "Planck scale" $L_{\rm P} \sim (g_s\alpha'^2)^{1/4}$),[4] then from the 1-d action or (3.7) we conclude that K and Q in (2.6) have dimension (length)^4 while P and f have dimension (length)^2. This should be kept in mind in what follows. It is easy to restore the dependence on the Planck length by rescaling $Q \to L_{\rm P}^4 Q$, $P \to L_{\rm P}^2 P$, etc. To restore the dependence on the string coupling one should further rescale $P^2 \to g_s P^2$. At the end, $Q \sim g_s\alpha'^2 N, P \sim g_s\alpha' M$.

As shown in [9, 12], there is a subclass of simple non-extremal solutions for which the functions K and z satisfy the same 1-st order equations as in the extremal case [8]. Indeed, if we set $K' = -2P^2e^{\Phi+4y+4w}$, then (3.6) implies that z should be subject to a 1-st order equation. In this case the 3-forms are self-dual. Then it is consistent, in particular, to keep $\Phi = 0$ and $w = 0$ so that $\mathbb{T}^{1,1}$ is not squashed. However, these simplest solutions turn out to be singular: they have a horizon coinciding with a curvature singularity [12].

To find a non-extremal generalization of the KT solution [8] with a *regular* horizon of the Schwarzschild type it is necessary to study the above second order system in its full generality. We shall be looking for a solution which satisfies the two natural requirements:

[3]The above system follows from the 1-d effective lagrangian $L = T - V$.

[4]Since we are concerned with gauge/gravity duality in this paper, $L_{\rm P}$ has nothing to do with the physical Planck length. Instead, due to the dimensional transmutation in this theory [4], $L_{\rm P}/P$ sets the scale of glueball masses.

(a) it is a one-parameter ($x' = a$ or Hawking temperature) generalization of the extremal KT solution;

(b) it reduces to the standard *regular* black D3-brane solution in the $P = 0$ limit.

Thus for $a \to 0$ the solution should reduce to the KT one [8]

$$x = w = \Phi = 0\,, \qquad e^{-4y} = 4u\,, \qquad K = -\frac{P^2}{2}\log(uL_{\rm P}^4)\,, \tag{3.8}$$

$$h = e^{-4z} = h_0 - \frac{P^2}{2}u\left[\log(uL_{\rm P}^4) - 1\right], \tag{3.9}$$

where $h_0 = 1$ for the asymptotically flat solution and $h_0 = 0$ for the analog of the near-horizon AdS part of the pure D3-brane background.

At the same time, for $P \to 0$ the solution should reduce to the regular black D3-brane background

$$\begin{aligned} w = \Phi = 0\,, \qquad e^{4x} = e^{4au}\,, \qquad e^{-4y} = a^{-1}\sinh 4au\,,\\ e^{-4z} = \frac{Q}{4a}\sinh 4a(u+k)\,. \end{aligned} \tag{3.10}$$

For the asymptotically flat ($h(0) = 1$) boundary conditions[5]

$$e^{4ak} = \gamma = Q^{-1}\left(\sqrt{Q^2 + 16a^2} + 4a\right), \tag{3.11}$$

the metric (2.3) should take the standard non-extremal D3-brane form [13, 14] with

$$\begin{aligned} g = \mathrm{g} = e^{-8x} = 1 - \frac{2a}{\rho^4}\,, \qquad \rho^4 = e^{4y+4x} = \frac{2a}{1 - e^{-8au}}\,,\\ h = e^{-4z-4x} = 1 + \frac{\tilde{Q}}{4\rho^4}\,, \qquad \tilde{Q} = \gamma^{-1}Q = \sqrt{Q^2 + 16a^2} - 4a\,. \end{aligned} \tag{3.12}$$

Note that near the horizon ($u \to \infty$):

$$\begin{aligned} y &= -au + \frac{\log 2a}{4} + \frac{e^{-8au}}{4} + O(e^{-16au})\,,\\ z &= -au + \frac{\log\frac{8a}{Q\gamma}}{4} + \frac{e^{-8au}}{4\gamma^2} + O(e^{-16au})\,, \end{aligned} \tag{3.13}$$

while at large distances ($u \to 0$)

$$\begin{aligned} y &= -\frac{1}{4}\log 4u - \frac{2}{3}a^2u^2 + O(u^3)\,,\\ e^{-4z} &= \frac{Q}{8a}(\gamma - \gamma^{-1}) + \frac{Q}{2}(\gamma + \gamma^{-1})u + O(u^2)\,, \end{aligned} \tag{3.14}$$

i.e.

$$e^{-4z} = 1 + \sqrt{Q^2 + a^2}\,u + O(u^2)\,, \qquad (e^{-4z})_{k=0} = Qu + O(u^2)\,. \tag{3.15}$$

[5] The choice of $k = 0$ in (3.10) leads to the black hole in AdS solution, where we drop the asymptotically flat region, i.e. $h(0) = 0$. In this case $\gamma = 1$, i.e. $\tilde{Q} = Q$ in the expression for h below.

4. Asymptotics of the regular non-extremal solution

Since we do not see a simple analytic solution to (3.2)–(3.6) which has the required properties, we need to outline a strategy for finding it in perturbation theory or numerically. This involves understanding the behavior of the solutions in the two asymptotic regions: $u \to \infty$ and $u \to 0$, i.e. in the short-distance and long-distance limits.

In order for the non-extremal generalization of the KT solution to reduce to the standard black D3-brane in the $P \to 0$ limit, the $u \to \infty$ asymptotics have to match (3.13):

$$x = au \,, \qquad y \longrightarrow -au + y_* \,, \qquad z \longrightarrow -au + z_* \,, \tag{4.1}$$

$$w \longrightarrow w_* \,, \qquad \Phi \longrightarrow \Phi_* \,, \qquad K \longrightarrow K_* \,. \tag{4.2}$$

The asymptotics (4.1) guarantee the existence of a regular Schwarzschild horizon at $u = \infty$, and it is natural to expect that w, K and Φ have stationary points at this horizon. Then it is easy to see that our system of eqs. (3.2)–(3.6) *and* the constraint (3.7) are indeed satisfied at large u. We will also show that turning on P makes a small perturbation on the large u asymptotics.

The information on the Hawking temperature is contained in the constants y_* and z_*. The natural near-horizon variable is $U = e^{-4au}$. Insisting that there is no conical singularity in the $U - X_0$ plane fixes the Hawking temperature to be

$$T = \frac{2}{\pi} a e^{2z_* - 5y_*} \,. \tag{4.3}$$

In general, there are different possibilities for the large u behaviour:

(i) The asymptotic $u \to \infty$ solution corresponds to the existence of a regular Schwarzschild horizon, i.e. a solution with the form (4.1).

(ii) e^{-4z} and thus $h = e^{-4z-4x}$ in (2.3) vanish at some *finite* u before we reach $u = \infty$; in this case there is a naked singularity which is not cloaked by a horizon.

(iii) It may also happen that $h = e^{-4z-4x}$ vanishes at $u = \infty$, so we get a horizon coinciding with the singularity as in the solution of [9].

We believe that possibility (ii) corresponds to $T < T_c$ where a U(1) symmetric solution is singular and one needs an appropriate KS-type ansatz [4] to remove the singularity. Possibility (i) corresponds to $T > T_c$ where our ansatz should be sufficient.

At large distances ($u \to 0$) the non-extremal solution should approach the extremal KT solution (3.8) and (3.9), i.e. we require that

$$u \longrightarrow 0 : \qquad x, w, \Phi \longrightarrow 0 \,, \qquad y \longrightarrow -\frac{1}{4} \log 4u \,. \tag{4.4}$$

Note that this behavior is also in agreement with the asymptotics (3.14) found for the regular D3-brane. The behaviors of the effective 3-brane charge (2.6) and of the warp factor are required to be

$$K(u) \longrightarrow -\frac{P^2}{2}\log(uL_{\rm P}^4)\,, \qquad e^{-4z} \longrightarrow -\frac{P^2}{2}u\log\left(uL_{\rm P}^4\right)\,, \tag{4.5}$$

where we adopt the choice $h_0 = 0$ such that there is no asymptotically flat region.

If we attempt integrating our 2nd-order equations from $u = 0$ starting with these asymptotics, we should vary the temperature (4.3) to find solutions where possibility (i) is realized and we reach the large u behavior (4.1).

As we integrate towards large u, the effective 3-brane charge $K(u)$ decreases: this is the varying flux phenomenon. For low temperature $K(u)$ reaches zero before the non-extremality effects have a chance to affect the solution significantly. This produces possibility (ii). However, we believe that there is a critical temperature $T_{\rm c}$ above which $K(u)$ is positive everywhere and possibility (i) is realized.

Thus, the task of the numerical work is to determine the accessible values of z_* and y_* as a function of the parameters, and to show that the allowed values of the temperature (4.3) are bounded from below in this symmetry restored phase.

If possibility (i) is indeed realized, then we expect the decreasing $K(u)$ to stabilize at a value K_* for large u. If the horizon has the Hawking temperature T, then in the dual field theory we interpret K_* as the effective number of unconfined color degrees of freedom at temperature T. As T approaches $T_{\rm c}$ from above, K_* should decrease. This would agree with the dual field theory interpretation: as we lower the temperature we excite a smaller effective number of colors. We believe that K_* should approach *zero* at the critical temperature $T_{\rm c}$. From the field theoretic point of view, $K_* \to 0$ because there is a reduction in the effective number of degrees of freedom at the phase transition.

5. Perturbation theory in P

One useful approach to constructing the required regular non-extremal solution is to start with the non-extremal D3-brane solution (3.10), which is valid for $P = 0$, and find its deformation order by order in P^2. A remarkable feature of perturbation theory in P^2 near the extremal ($a = 0$) D3-brane background is that already the *first-order* correction gives the *exact* form of the KT solution (3.8) and (3.9). Therefore, it is natural to expect that a similar expansion near the non-extremal D3-brane solution will capture the basic features of non-extremal generalization of the KT background.

More precisely, our starting point will be the well-known D3-brane solution (3.10) with Q replaced by the effective 3-brane charge K_*, so that we automatically match onto the near-horizon asymptotics (4.1) and (4.2). Perturbing in P^2 around the near-extremal 3-brane solution of charge K_*, we will see that the $O(P^2)$ modification is already enough to match onto the KT long-distance asymptotics. The small parameter governing this expansion is actually the dimensionless ratio $P^2K_*^{-1}$,[6] i.e. for this method to work the horizon value of the effective 3-brane charge K_* has to be sufficiently large. In view of the discussion in section 4, this means that this perturbation theory is applicable for $T \gg T_c$.

It is useful to rescale the fields by appropriate powers of P^2, setting

$$K(u) = K_* + 2P^2F(u)\,, \qquad \Phi(u) = P^2\phi(u)\,, \qquad w(u) = P^2\omega(u) \tag{5.1}$$

and

$$\begin{aligned} &y \longrightarrow y + P^2\xi\,, \qquad e^{-4z} \longrightarrow e^{-4z} + P^2\zeta\,, \\ \text{i.e.}\qquad &z \longrightarrow z + P^2\eta\,, \qquad \zeta = -4e^{-4z}\eta + O(P^2)\,, \end{aligned} \tag{5.2}$$

where y, z represent the pure D3-brane solution (3.9): $e^{-4y} = a^{-1}\sinh 4au, e^{-4z} = \frac{K_*}{4a}\sinh 4a(u+k)$, and ξ and ζ or η are corrections to it. To match onto the small u KT asymptotics, $e^{-4z} \to -\frac{P^2}{2}u\log(uL_{\mathrm{P}}^4)$, we require that

$$\omega(0) = \xi(0) = \phi(0) = 0\,, \qquad F \longrightarrow -\frac{1}{4}\log u\,, \qquad \zeta \longrightarrow -\frac{1}{2}u\log(8au)\,. \tag{5.3}$$

For $k = 0$, i.e. for the case without an asymptotically flat region, this means

$$\eta \longrightarrow \frac{1}{8K_*}\log(8au)\,. \tag{5.4}$$

Note that η is not uniformly small and it seems better to consider the expansion in terms of ζ which goes to zero. Moreover, the leading-order correction to ζ already reproduces the exact KT solution. However, ζ and η are directly related, so the two expansions are, in fact, equivalent and we will find it more convenient to use η.

Now the system (3.2)–(3.6) takes the following explicit form:

$$10\xi'' - 320e^{8y}\xi + \phi'' + O(P^2) = 0\,, \tag{5.5}$$

$$10\omega'' - 120e^{8y}\omega - \phi'' + O(P^2) = 0\,, \tag{5.6}$$

$$\phi'' + e^{4z-4y}\left(F'^2 - e^{8y}\right) + O(P^2) = 0\,, \tag{5.7}$$

$$\left(e^{4z-4y}F'\right)' - K_*e^{8z} + O(P^2) = 0\,. \tag{5.8}$$

$$4\eta'' - 8K_*^2e^{8z}\eta - 4K_*Fe^{8z} - e^{4z-4y}(F'^2 + e^{8y}) + O(P^2) = 0\,. \tag{5.9}$$

The constraint (3.7) becomes

$$10y'\xi' - 4z'\eta' - \frac{1}{4}e^{4z-4y}F'^2 - 40e^{8y}\xi + \frac{1}{4}e^{4z+4y} + K_*^2e^{8z}\eta + \frac{1}{2}K_*e^{8z}F + O(P^2) = 0\,. \tag{5.10}$$

[6] Note that $P \sim g_sM$ and $K \sim g_sN$ where M and N are the numbers of fractional and regular D3-branes respectively.

5.1 Leading-order solution for K

Using (3.5), i.e. $K_* e^{8z} = 4K_*^{-1} z''$, we get from (5.8) and (3.10)

$$F' = c - K_*^{-1} e^{4y} (e^{-4z})' = c - \frac{a \cosh 4a(u+k)}{\sinh 4au} . \tag{5.11}$$

For large u (near the horizon), we must have $F' \to 0$ in order to satisfy (4.2). This fixes the integration constant to be $c = a\gamma = ae^{4ak} = a\gamma$, so that

$$F' = -\frac{2a\beta}{e^{8au} - 1} , \qquad \beta = \cosh 4ak = \left(1 + \frac{16a^2}{K_*^2}\right)^{1/2} , \tag{5.12}$$

and thus

$$F = -\frac{1}{4}\beta \log(1 - e^{-8au}) . \tag{5.13}$$

As required by (5.1), this expression satisfies $F(u \to \infty) \equiv F_* = 0$.

We will be particularly interested in the case $k = 0$ in (3.10), when the starting point of the perturbation theory has AdS asymptotics rather than joining onto asymptotically flat space. In this limit $\beta = 1$ in (5.12)[7] and thus

$$K(u) = K_* - \frac{P^2}{2} \log(1 - e^{-8au}) . \tag{5.14}$$

This expression approaches K_* for large u, as required. If $P^2 K_*^{-1} \ll 1$ then the second term is a small perturbation for almost all values of u, except close to zero. Thus, it seems that our perturbation theory breaks down near $u = 0$. Nevertheless, we note that (5.14) has precisely the same small u asymptotics (4.5) as the extremal KT solution!

Thus, already at the leading order this perturbation theory produces a solution with the correct KT asymptotics. This remarkable fact strengthens our confidence that an exact solution interpolating between the KT solution at small u and the regular D3-brane horizon at large u indeed exists. Our perturbed solution should be a good approximation to it provided that $P^2 K_*^{-1} \ll 1$. This limit corresponds to high Hawking temperatures. To show this, let us match (5.14) with (4.5) for small u. We find that

$$8aL_{\rm P}^{-4} = e^{2/\lambda} , \qquad \lambda \equiv P^2 K_*^{-1} \ll 1 . \tag{5.15}$$

On the other hand, the Hawking temperature is determined in terms of the non-extremality a and the charge near horizon K_* by the usual near-extremal D3-brane formula (cf. (4.3) and (3.10), with $Q \to K_*$)

$$T \sim a^{1/4} K_*^{-1/2} . \tag{5.16}$$

[7] In the $k = 0$ case the r.h.s. of the expressions in (3.11) and (5.12) do not, of course, apply.

We should express the temperature in terms of the glueball mass scale [4]

$$\Lambda = L_{\rm P} P^{-1} , \tag{5.17}$$

which we also expect to be the scale of the critical temperature: $T_{\rm c} \sim \Lambda$. Using (5.17) and (5.15) we find

$$T \sim \Lambda \sqrt{\lambda} e^{1/(2\lambda)} \sim T_{\rm c} \sqrt{\lambda} e^{1/(2\lambda)} . \tag{5.18}$$

Thus, in our perturbative regime we find $T \gg T_{\rm c}$; as expected, this regime is applicable far above the phase transition into the chiral symmetry restored phase.

5.2 Solutions for other fields

Let us now solve for perturbations of other fields. Using (3.10) and (5.12) the equation for the dilaton (5.7) becomes

$$\phi'' = \frac{4a^2}{K_*} \frac{1 - e^{-8au} \cosh^2 4ak}{\sinh 4a(u+k) \sinh 4au} . \tag{5.19}$$

Expanding in powers of u this gives $\phi'' = -\frac{4a^2}{K_*^2 u} + c_1 + c_2 u + \cdots$, so that the small u asymptotics of ϕ is $u \log u$. To simplify the expressions, let us consider again the case of $k = 0$. Then

$$\phi'' = \frac{16a^2}{K_*(e^{8au} - 1)} , \qquad \text{i.e.} \qquad \phi' = \frac{2a}{2K_*} \log(1 - e^{-8au}) , \tag{5.20}$$

where we have fixed the integration constant so that $\phi' \to 0$ at $u \to \infty$,

$$\phi = \phi_* + \frac{1}{4K_*} \operatorname{Li}_2(e^{-8au}) , \qquad \phi_* = -\frac{\pi^2}{24K_*} . \tag{5.21}$$

$\operatorname{Li}_n(z)$ denotes the polylogarithm function. The limits of this expression are

$$u \longrightarrow 0 : \ \phi = \frac{2a}{K_*} \left[u(\log 8au - 1) + O(u^2 \log u) \right] , \tag{5.22}$$

$$u \longrightarrow \infty : \ \phi = \phi_* + \frac{1}{4K_*} e^{-8au} + O(e^{-16au}) . \tag{5.23}$$

This implies that the string coupling e^{Φ} decreases as we approach the horizon.

Next, we are to find ω, ξ, η satisfying (5.5), (5.6) and (5.9), i.e. (for $k = 0$)

$$\omega'' - \frac{12a^2}{\sinh^2 4au} \omega = \frac{8a^2}{5K_*(e^{8au} - 1)} , \tag{5.24}$$

$$\xi'' - \frac{32a^2}{\sinh^2 4au} \xi = -\frac{8a^2}{5K_*(e^{8au} - 1)} , \tag{5.25}$$

$$\eta'' - \frac{32a^2}{\sinh^2 4au} \eta = \frac{a^2}{K_* \sinh^2 4au} \left[1 + e^{-8au} - 4 \log(1 - e^{-8au}) \right] . \tag{5.26}$$

To analyze these equations for $a \neq 0$ it is convenient to introduce a new radial variable

$$v = 1 - e^{-8au}\,. \tag{5.27}$$

Then (5.24), (5.25), and (5.26) can be expressed in the forms

$$\begin{aligned}
v(1-v)\omega'' - v\omega' - \frac{3/4}{v}\omega &= \frac{1}{40K_*}\,, \\
v(1-v)\xi'' - v\xi' - \frac{2}{v}\xi &= -\frac{1}{40K_*}\,, \\
v(1-v)\eta'' - v\eta' - \frac{2}{v}\eta &= \frac{1}{16K_* v}(2 - v - 4\log v)\,,
\end{aligned} \tag{5.28}$$

where primes now denote d/dv. Now, the homogenous equation

$$v(1-v)f'' - vf' - \frac{A}{v}f = 0$$

is solved for generic A by $f(v) = v^\nu\,{}_2F_1(\nu,\nu;2\nu;v)$, where ${}_2F_1$ is the hypergeometric function and $\nu(\nu-1) = A$. As it happens, $A = 2$ is a degenerate case where the solutions to the homogenous equation are elementary functions of v (namely, $1/v - 1/2$ and $-2 + \frac{v-2}{v}\log(1-v)$). Using these solutions one can extract solutions to the inhomogenous equations as well:

$$\begin{aligned}
\xi &= \frac{2v + [-2v + (v-2)\log(1-v)]\log v + (v-2)\,\mathrm{Li}_2(v)}{40K_* v}\,, \\
\eta &= \frac{v-2}{16K_* v}\left[\log v \log(1-v) + \mathrm{Li}_2(v)\right].
\end{aligned} \tag{5.29}$$

For $v \to 0$, which corresponds to $u \to 0$, we have the asymptotics

$$\xi \sim \frac{v}{80K_*} + O\left(v^2 \log v\right), \tag{5.30}$$

$$\eta \sim \frac{\log v - 1}{8K_*} + \frac{v}{32K_*} + O\left(v^2 \log v\right). \tag{5.31}$$

This changes the AdS asymptotics into the KT asymptotics in agreement with (4.4).

For $v \to 1$, which corresponds to $u \to \infty$, we have the near-horizon asymptotics consistent with our expectations (4.1) and (4.2):

$$\xi \sim \frac{12-\pi^2}{240K_*} + \frac{9-\pi^2}{120K_*}(1-v) + O\left[(1-v)^2\right], \tag{5.32}$$

$$\eta \sim -\frac{\pi^2}{96K_*} + \frac{3-\pi^2}{48K_*}(1-v) + O\left[(1-v)^2\right]. \tag{5.33}$$

To summarize, the horizon values of the perturbations we have solved for are

$$\xi_* = \frac{12-\pi^2}{240K_*}\,, \qquad \eta_* = -\frac{\pi^2}{96K_*}\,, \qquad \phi_* = -\frac{\pi^2}{24K_*}\,. \tag{5.34}$$

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The "squash factor" ω in (2.2) is special in that the volume of compact space does not depend on it. Hence it cancels from the observables like the horizon area and the Hawking temperature. An explicit expression for ω is also available in principle, but it involves some messy integrals, so we will just determine its asymptotics schematically. At large u (5.24) becomes

$$\omega'' - 48a^2 e^{-8au}\omega = \frac{8a^2}{5K_*}e^{-8au}\,, \tag{5.35}$$

and a solution of this equation is given in terms of Bessel functions of e^{-8au} so that the asymptotics is

$$\omega = \omega_* + \omega_1 e^{-8au} + O(e^{-16au}) + \cdots\,, \qquad \omega_1 = \frac{3}{4}\omega_* + \frac{1}{40K_*}\,. \tag{5.36}$$

For small u (again, in the $k = 0$ case) we get

$$\omega'' - \frac{3}{4u^2}\omega = \frac{a}{5K_* u}\,, \tag{5.37}$$

with the general solution satisfying (4.4) being

$$u \longrightarrow 0: \qquad \omega = -\frac{4a}{15K_*}u + b_1 u^{3/2}\,. \tag{5.38}$$

Entropy and temperature can be determined straightforwardly now that the metric is known. The reader will have no trouble verifying that the metric (2.1) can be cast into the form

$$\begin{aligned} ds^2_{10E} = {} & \frac{\sqrt{8a/K_*}}{\sqrt{v}}e^{2P^2\eta}\left[(1-v)dX_0^2 + dX_i^2\right] + \frac{\sqrt{K_*}}{32}e^{-2P^2(\eta-5\xi)}\frac{dv^2}{v^2(1-v)} + \\ & + \frac{\sqrt{K_*}}{2}e^{-2P^2(\eta-\xi)}\left[e^{-8P^2\omega}e_\psi^2 + e^{2P^2\omega}\left(e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2\right)\right]. \end{aligned} \tag{5.39}$$

Using the explicit formulas for ξ (5.30) and η (5.31), we obtain an explicit expression for the entropy per unit volume divided by the temperature cubed:

$$\frac{S}{VT^3} = \alpha\frac{K_*^2}{L_{\rm P}^8}e^{4P^2(5\xi_*-2\eta_*)} = \alpha\frac{K_*^2}{L_{\rm P}^8}\left[1 + \lambda + O(\lambda^2)\right], \tag{5.40}$$

where α is a factor of order unity which can be fixed by specializing to the $P = 0$ case and using the results of [10]. Unravelling the various definitions that we have made, one can show that $K_*/L_{\rm P}^4$ is the flux of the five-form through $\mathbb{T}^{1,1}$, measured in Dirac units. This is the effective number of colors available at a given scale. (As an aside, it is gratifying to observe that the many factors of 2, 3, and π cancel out to give a simple 1 as the coefficient of $\lambda = P^2/K_*$ in the function in square brackets.)

The formula (5.40) does not identify how K_* depends on T. To determine this, one must be careful in defining exactly what temperature means in this non-asymptotically flat geometry. Equivalently, we need to specify a way of normalizing time which is invariant under changes of a. A sensible procedure is to pick some very large K_0 (much larger than K_* for the range of a we are considering) and then define a privileged time coordinate τ by the condition that $g_{\tau\tau} = 1$ at the radius where when $K(u) = K_0$. One may easily verify from (5.39) that the coordinate we have called X_0 is *not* such a coordinate. But it is equivalent to compute the Hawking temperature from (5.39) in the usual way (by eliminating the conical deficit at $v = 1$) and then multiply by the appropriate power of g_{00} to convert to what one would have found using τ. The end result is precisely (5.18), which can be inverted to read

$$\frac{K_*}{2P^2} = \log\frac{T}{\Lambda} + \frac{1}{2}\log\log\frac{T}{\Lambda} + \cdots . \tag{5.41}$$

Recall that Λ is the scale of glueball masses. Neglecting the $\log\log$ correction, we recover a result expected by Buchel [9]:

$$\frac{S}{VT^3} \sim \frac{P^4}{L_{\mathrm{P}}^8}\left(\log\frac{T}{\Lambda}\right)^2 . \tag{5.42}$$

Physically speaking, we discover that the effective number of colors at a given temperature T rises as $\log(T/\Lambda)$. This is consistent with the finding [15] that two-point functions, computed from supergravity, scale as a negative power of the separation x times $(\log x\Lambda)^2$.

On the supergravity side, it seems reasonable to suppose that the critical solution has $K_* \to 0$ (that is, zero F_5 at the horizon), since this is where, as suggested by (5.40), the entropy must go to zero — and we believe it must do so in order for correlators to have power law tails. All this is consistent with the view that chiral symmetry breaking occurs simultaneously with confinement in the gauge theory.

6. Concluding remarks

We have carried out a perturbation expansion in P to the leading non-trivial order for all the fields involved in the non-extremal KT solution. The real expansion parameter turns out to be $\lambda = P^2/K_*$. Since K_* is the five-form flux at the horizon, and this quantity gets bigger and bigger as we push the horizon further into the UV, it is clear that the small λ expansion is precisely a high temperature expansion (see (5.18)). At the first non-trivial order in perturbation theory, we find a smooth interpolation between the KT solution and the near-extremal D3-brane — or, to put it differently, a regular non-extremal generalization of the KT geometry. It is necessary for the U(1) fiber of $\mathbb{T}^{1,1}$ to become squashed, and for the dilaton to run. There is a mild pathology in the perturbation theory, in that both F and η

in (5.1) and (5.2) acquired $\log u$ divergences in the UV; but these are precisely the divergences needed to match onto the KT solution. The perturbation fields approach constant values at the IR horizon, which guarantees that this is a regular horizon, not a singularity. Further terms in the perturbation series should be *uniformly* small for all fields. We conclude that the supergravity dual of the $\mathrm{SU}(N+M)\times\mathrm{SU}(N)$ gauge theory at very high temperature is described by a background containing a well-developed Schwarzschild horizon in an asymptotically KT geometry, as suggested in [9]. This background is U(1) symmetric; therefore, the chiral symmetry is indeed restored at high enough temperature.

As we increase the value of $\lambda = P^2 K_*^{-1}$, which corresponds to decreasing the temperature, the perturbation theory becomes less reliable. The obvious alternative is numerical integration. The main difficulty here is that some of the boundary conditions on the second order differential equations are set at the horizon, while the others are set in the UV by requiring KT asymptotics. This makes "shooting" algorithms tedious since one is operating in a multi-dimensional space of boundary conditions. It is amusing that the perturbative approach circumvents this difficulty, and it is possible that some hybrid of a perturbative treatment and numerics might be developed. There is a good motivation for studying the temperature dependence of the background: it might be possible to approach the chiral symmetry breaking point $T = T_{\mathrm{c}}$ and characterize the dual of the critical theory.

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A. Asymptotic expansions of the solution

In this appendix we consider the asymptotic solutions of our system in the near-horizon (large u) and long-distance (small u) limits. In the absence of exact solutions, this a necessary preliminary to any approximation scheme, numerical or otherwise, for obtaining a complete solution.

To gain some intuition about the role of w we shall start in section A.1 with an analysis of the solution in the absence of any matter fields. Thus only y'' and w'' will be non zero. Then in section A.2 we shall treat the full system of equations. To simplify notation, we work in units where $L_{\mathrm{P}} = 1$.

A.1 Solutions for y and w in the absence of matter

Assuming $Q = P = f = \Phi = 0$ and choosing x and z to satisfy (4.1), the 2nd order system (3.2)–(3.7) reduces to

$$x' = a\,, \qquad z' = -a\,, \tag{A.1}$$

$$y'' = \frac{4}{5}e^{8y}\left(6e^{-2w} - e^{-12w}\right),$$
$$w'' = \frac{6}{5}e^{8y}\left(e^{-2w} - e^{-12w}\right), \tag{A.2}$$

$$y'^2 - w'^2 - \frac{1}{5}e^{8y}\left(6e^{-2w} - e^{-12w}\right) = a^2\,. \tag{A.3}$$

If the constraint (A.3) is satisfied at one point $u = u_0$, it is satisfied for all u because of the equations of motion (A.2). We can satisfy the constraint at $u \to \infty$ by choosing the asymptotics to be as in (4.1)

$$y' \longrightarrow -a\,, \qquad w' \longrightarrow 0\,. \tag{A.4}$$

Let us first set $a = 0$ and consider the extremal BPS solution described by the 1-st order system $y' + \frac{1}{5}e^{4y}(3e^{4w} + 2e^{-6w}) = 0$, $w' - \frac{3}{5}e^{4y}(e^{4w} - e^{-6w}) = 0$ (see [12, eq. (3.11)] and also [8, 16]). From the equation for dy/dw one finds that

$$y = -\int dw \frac{3e^{10w} + 2}{3(e^{10w} - 1)} = y_0 - \frac{1}{6}\log\left(e^{6w} - e^{-4w}\right), \tag{A.5}$$

i.e.

$$e^{-6(y-y_0)} = e^{6w} - e^{-4w}\,. \tag{A.6}$$

That gives $w' = \frac{3}{5}e^{4y_0}(1 - e^{-10w})^{1/3}$, implying that w can be expressed implicitly, $u = u(w)$, in terms of a sum of logarithms and arctangents. We set the integration constant by requiring that $w = 0$ when $u = 0$. Introducing $\rho = e^{y+w}$, the resulting Ricci flat 6-d metric is then the generalized conifold of [17]

$$ds_6^2 = \kappa^{-1}(\rho)d\rho^2 + \rho^2\left[\kappa(\rho)e_\psi^2 + e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2\right],$$
$$\kappa(\rho) = e^{-10w} = 1 - \frac{\rho_*^6}{\rho^6}\,, \qquad \rho = e^{y+w}\,, \qquad \rho_* = e^{y_0} \le \rho < \infty\,. \tag{A.7}$$

w changes from 0 at $\rho = \infty$ ($u = 0$) to ∞ at $\rho = \rho_*$ ($u = \infty$), while $y = \infty$ at $\rho = \infty$ and $y = -\infty$ at $\rho = \rho_*$. The asymptotics near $\rho = \rho_*$, i.e. $u \to \infty$ are

$$y = -cu + y_* + y_1 e^{-10cu} + \cdots\,, \qquad w = cu + w_* + w_1 e^{-10cu} + \cdots\,, \tag{A.8}$$

where

$$c = \frac{3}{5}e^{4y_0}\,, \qquad y_* = y_0 - w_*\,, \qquad w_* = \frac{\sqrt{3}\pi}{60} - \frac{3}{20}\log 3\,, \tag{A.9}$$

and $y_1 = \frac{2}{15}e^{-10w_*}, w_1 = \frac{1}{30}e^{-10w_*}$. To find the "non-extremal" analog of this space, we are to switch on $a \neq 0$. Then the system (A.2) and (A.3) does not have a simple analytic solution, but it is easy to study the asymptotics at small and large u. The long-distance asymptotic solution is

$$u \longrightarrow 0: \ y = -\frac{1}{4}\log 4u - \frac{2}{3}a^2u^2 - \frac{3}{4}s^2u^3 + \frac{16}{45}a^4u^4 + \frac{16}{11}s^3u^{9/2} + \cdots,$$
$$w = su^{3/2} - s^2u^3 - \frac{1}{2}a^2su^{7/2} + \frac{11}{6}s^3u^{9/2} + \cdots. \quad \text{(A.10)}$$

This is consistent with the extremal generalized conifold solution (A.7) in the $a \to 0$ limit if $s = \frac{4}{5}\rho_*^6$. Taking instead s to be proportional to a we recover the conifold solution with $w = 0$ in the $a \to 0$ limit.

The short-distance asymptotics is described by

$$u \longrightarrow \infty: \ y = -bu + y_* + y_1^{(1)}e^{-(8b+2\nu)u} + y_1^{(2)}e^{-(8b+12\nu)u} + O(e^{-2(8b+2\nu)u}), \quad \text{(A.11)}$$
$$w = \nu u + w_* + w_1^{(1)}e^{-(8b+2\nu)u} + w_1^{(2)}e^{-(8b+12\nu)u} + O(e^{-2(8b+2\nu)u}), \quad \text{(A.12)}$$

where

$$b^2 - \nu^2 = a^2, \qquad y_1^{(1)} = \frac{24e^{8y_*-2w_*}}{5(8b+2\nu)^2}, \qquad w_1^{(1)} = \frac{6e^{8y_*-2w_*}}{5(8b+2\nu)^2},$$
$$y_1^{(2)} = -\frac{4e^{8y_*-12w_*}}{5(8b+12\nu)^2}, \qquad w_1^{(2)} = -\frac{6e^{8y_*-12w_*}}{5(8b+12\nu)^2}. \quad \text{(A.13)}$$

These asymptotics reduce to the extremal solution (A.8) in the limit $a \to 0$ provided we set $b = c$.

A numerical analysis confirms the existence of a solution of (A.1)–(A.3) interpolating between the two asymptotic regions (A.10), (A.11) and (A.12).

To have a solution reducing to the standard ($w = 0$) conifold one in the $a \to 0$ limit, we should set[8]

$$\nu = na, \qquad b = (1+n^2)^{1/2}a, \qquad e^{4y_0} = \frac{5}{3}b. \quad \text{(A.14)}$$

To satisfy (A.4) we are to assume the simplest possibility – $n = 0, \nu = 0$, i.e. $b = a$. For $\nu = 0$ the function w goes to a constant instead of infinity and the asymtotic $u \to \infty$ solution is found to be

$$u \longrightarrow \infty: \ y = -au + y_* + y_1e^{-8au} + \cdots,$$
$$y_1 = \frac{1}{80a^2}e^{8y_*}\left(6e^{-2w_*} - e^{-12w_*}\right), \quad \text{(A.15)}$$
$$w = w_* + w_1e^{-8au} + \cdots, \qquad w_1 = \frac{3}{160a^2}e^{8y_*}\left(e^{-2w_*} - e^{-12w_*}\right). \quad \text{(A.16)}$$

[8]One is to resum the series in (A.11) in order to reproduce the exact conifold solution $y = -\frac{1}{4}\log 4u$.

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The behaviour of w follows from the equation $w'' = 12e^{8y_* - 8au}w + O(w^2)$. Once we include charges, this equation (i.e. (3.3)) will have an extra inhomogeneous $O(a^2P^2)$ term (cf. (5.6) and (5.20)). The variation of w should be driven only by the non-extremality *and* non-zero P, so that w should vanish in any of the limits $a \to 0$ or $P \to 0$.

For finite a the point $u = \infty$ is the horizon. For general ν (i.e. b not necessarily equal to a) the near-horizon metric (2.1) becomes (using (A.1), (A.11) and (A.12)):

$$ds^2_{10E} \longrightarrow e^{2z_*}(e^{-8au}dX_0^2 + dX_i dX_i) + e^{10y_* - 2z_* - 10bu + 2au}du^2 + \\ + e^{2y_* - 2z_* - 2bu + 2au}\left[e^{-8w_* - 8\nu u}e_\psi^2 + e^{2w_* + 2\nu u}(e_{\theta_1}^2 + e_{\phi_1}^2 + e_{\theta_2}^2 + e_{\phi_2}^2)\right]. \quad \text{(A.17)}$$

While the Ricci tensor of this metric vanishes at large u, the full curvature invariants are singular at $u = \infty$ for generic values of ν.[9]

The *only* case when the horizon is *regular* is precisely the one we are interested in: $\nu = 0,\ b = a$. In this case (A.17) is indeed the asymptotic form of this "black hole on conifold", i.e. the zero-charge ($q = 0$, $h = 1$) case of the regular non-extremal D3-brane metric (3.12). In this case (A.17) factorizes into $\mathbb{R}^2_{u,X_0} \times \mathbb{R}^3_{X_i} \times \mathbb{T}^{1,1}$ with the flat (u, X_0) part being

$$ds_2^2 = \frac{e^{10y_* - 2z_*}}{16a^2}(dU^2 + U^2 d\tilde{X}_0^2), \qquad U = e^{-4au}, \qquad \tilde{X}_0 = 4ae^{2z_* - 5y_*}X_0. \quad \text{(A.18)}$$

As usual, X_0 is to be made periodic to avoid the conical singularity, thus determining the Hawking temperature (4.3) of this Schwarzschild 7-d black hole.

It is this $b = a, \nu = 0$ solution with non-vanishing w interpolating between (A.10) and (A.15) that we would like now to generalize to the fractional D3-brane case.

A.2 Asymptotic solutions for non-zero charges

We shall proceed by determining the asymptotic $u \to \infty$ and $u \to 0$ solutions of the system (3.2)–(3.7), now for non-vanishing charges. It is the non-extremality parameter a that should drive the variation of w and Φ, so we should choose the integration constants appropriately.

Let us start with the near-horizon region. Generalizing (4.1), (A.15) and (A.16) we shall set

$$u \longrightarrow \infty: \ x = au, \qquad y \longrightarrow -au + y_* + y_1 e^{-8au} + \cdots, \\ z \longrightarrow -au + z_* + z_1 e^{-8au} + \cdots, \qquad w \longrightarrow w_* + w_1 e^{-8au} + \cdots, \\ \Phi \longrightarrow \Phi_* + \Phi_1 e^{-8au} + \cdots, \qquad K \longrightarrow K_* + 2Pf_1 e^{-8au} + \cdots, \quad \text{(A.19)}$$

where the expansion goes in powers of e^{-8au}. These asymptotics are justified also

[9] For example, choosing $\nu = a$ we get the leading singularity as $(R_{...})^2 \sim e^{\kappa u}$, where $\kappa = 4(5\sqrt{2} - 1) > 0$.

by the results of perturbation theory in P starting with the near-horizon region of the standard non-extremal D3-brane as described in section 5. (Note that K_* is the same as in (5.1) up to terms of $O(P^2)$.)

It is straightforward to check that this ansatz is consistent with our full system of eqs. (3.2)–(3.7), determining the coefficients as follows (cf. (A.15) and (A.16)):

$$
\begin{aligned}
640a^2 y_1 &= 8e^{8y_*}(6e^{-2w_*} - e^{-12w_*}) - P^2 e^{\Phi_* + 4y_* + 4z_* + 4w_*}\,, \\
640a^2 w_1 &= 12e^{8y_*}(e^{-2w_*} - e^{-12w_*}) + P^2 e^{\Phi_* + 4y_* + 4z_* + 4w_*}\,, \\
256a^2 z_1 &= K_*^2 e^{8z_*} + P^2 e^{\Phi_* + 4y_* + 4z_* + 4w_*}\,, \\
64a^2 f_1 &= PK_* e^{\Phi_* + 4y_* + 4z_* + 4w_*}\,, \qquad 64a^2 \Phi_1 = P^2 e^{\Phi_* + 4y_* + 4z_* + 4w_*}\,.
\end{aligned} \tag{A.20}
$$

In order to satisfy the requirement of correspondence with the D3-brane solution in the $P \to 0$ limit we may choose

$$
\begin{aligned}
e^{4y_*} &= 2ae^{4P^2\xi_*}\,, \qquad e^{4z_*} = \frac{8a}{K_*} e^{4P^2\eta_*}\,, \\
e^{w_*} &= e^{P^2\omega_*}\,, \qquad e^{\Phi_*} = e^{P^2\phi_*}\,.
\end{aligned} \tag{A.21}
$$

Then, to $O(P^2)$, (A.20) become

$$
\begin{aligned}
y_1 &= \frac{1}{4} + \frac{P^2}{120K_*}(9 - \pi^2)\,, \qquad w_1 = \frac{P^2}{40K_*}(1 + 30\omega_* K_*)\,, \\
z_1 &= \frac{1}{4} + \frac{P^2}{48K_*}(3 - \pi^2)\,, \qquad f_1 = \frac{P}{4}\,, \qquad \Phi_1 = \frac{P^2}{4K_*}\,.
\end{aligned} \tag{A.22}
$$

The values of these coefficients are in agreement with the large u limits of the leading-order in P corrections to our fields found in section 5 (see (5.14), (5.23), (5.32), (5.33) and (5.36)).

Next, we need to fix the asymptotic behaviour at $u \to 0$. We should reproduce the black D3-brane asymptotics (3.14), (3.15) for $P = 0$, and the KT asymptotics (3.8) and (3.9) for $a = 0$. It is therefore reasonable to assume that the first terms in the asymptotic expansion are (in the $k = 0$ case in (3.15))

$$
\begin{aligned}
y &= -\frac{1}{4}\log 4u + \cdots\,, \qquad e^{-4z} = u(\mathrm{z}_{1,0} + \mathrm{z}_{1,1}\log u) + \cdots\,, \\
K &= -\frac{P^2}{2}\log u + \cdots\,.
\end{aligned} \tag{A.23}
$$

Moreover, we expect w and Φ to vanish at $u = 0$. It turns out to be easier to find expansions for $\exp(-4w)$, $\exp(-2w)$, and $\exp(-\Phi)$ than for their corresponding logarithms. We have found a small u expansion of the following form (assuming again that the asymptotically flat region is omitted, i.e. $k = 0$):

$$
\begin{aligned}
e^{-4y} &= 4u + \mathrm{y}_1 u^2 + u^3\left(\mathrm{y}_{2,0} + \mathrm{y}_{2,1}\log u + \mathrm{y}_{2,2}(\log u)^2\right) + \mathrm{y}_3 u^{7/2} + O(u^4)\,, && \text{(A.24)}\\
e^{-4z} &= u(\mathrm{z}_{1,0} + \mathrm{z}_{1,1}\log u) + u^2(\mathrm{z}_{2,0} + \mathrm{z}_{2,1}\log u) + \mathrm{z}_3 u^{5/2} + \\
&\quad + u^3\left(\mathrm{z}_{4,0} + \mathrm{z}_{4,1}\log u + \mathrm{z}_{4,2}(\log u)^2 + \mathrm{z}_{4,3}(\log u)^3\right) + \\
&\quad + \mathrm{z}_5 u^{7/2} + O(u^4)\,, && \text{(A.25)}\\
K &= -\frac{1}{2}P^2\log u + 2P\Big(u(\mathrm{f}_{1,0} + \mathrm{f}_{1,1}\log u) + \mathrm{f}_2 u^{3/2} + \\
&\qquad + u^2\left(\mathrm{f}_{3,0} + \mathrm{f}_{3,1}\log u + \mathrm{f}_{3,2}(\log u)^2\right) + \\
&\qquad + u^{5/2}(\mathrm{f}_{4,0} + \mathrm{f}_{4,1}\log u) + O(u^3)\Big)\,, && \text{(A.26)}\\
e^{-2w} &= 1 + \mathrm{w}_1 u + \mathrm{w}_2 u^{3/2} + u^2(\mathrm{w}_{3,0} + \mathrm{w}_{3,1}\log u) + \mathrm{w}_4 u^{5/2} + O(u^3)\,, && \text{(A.27)}\\
e^{-\Phi} &= 1 + u(p_{1,0} + p_{1,1}\log u) + u^2\left(p_{2,0} + p_{2,1}\log u + p_{2,2}(\log u)^2\right) + \\
&\quad + p_3 u^{5/2} + O(u^3)\,, && \text{(A.28)}
\end{aligned}
$$

It is easy to write the relations between the first few coefficients:

$$
\mathrm{z}_{1,1} = -\frac{1}{2}P^2\,, \qquad \mathrm{z}_{1,0} = \frac{1}{2}P^2\,, \qquad p_{1,1} = \frac{5}{4}\mathrm{y}_1\,, \qquad \mathrm{w}_1 = -\frac{1}{3}\mathrm{y}_1\,. \qquad \text{(A.29)}
$$

Certain other coefficients are also relatively simple:

$$
\begin{aligned}
\mathrm{y}_{2,2} &= \frac{5}{6}\mathrm{y}_1^2\,, \qquad \mathrm{y}_3 = -\frac{48}{7P}\mathrm{y}_1\mathrm{f}_2\,, \qquad \mathrm{z}_{2,1} = \frac{5}{8}P^2\mathrm{y}_1\,, \qquad \mathrm{z}_3 = \frac{4}{5}P\mathrm{f}_2\,,\\
\mathrm{z}_{4,3} &= -\frac{25}{192}P^2\mathrm{y}_1^2\,, \qquad \mathrm{f}_{1,1} = \frac{5}{8}P\mathrm{y}_1\,, \qquad \mathrm{f}_{3,2} = -\frac{35}{96}P\mathrm{y}_1^2\,, \qquad \mathrm{w}_2 = \frac{3}{P}\mathrm{f}_2\,,\\
\mathrm{w}_{3,1} &= -\frac{1}{2}\mathrm{y}_1^2\,, \qquad \mathrm{w}_4 = -\frac{29}{8P}\mathrm{y}_1\mathrm{f}_2\,, \qquad p_{2,2} = \frac{25}{32}\mathrm{y}_1^2\,, \qquad p_3 = -\frac{2}{P}\mathrm{y}_1\mathrm{f}_2\,. \qquad \text{(A.30)}
\end{aligned}
$$

The remaining coefficients are more complicated, and we will not write them down. All of the coefficients can be expressed in terms of y_1, $\mathrm{f}_{1,0}$, f_2, y_3, and $\mathrm{f}_{3,0}$. In other words, these five coefficients correspond to five undetermined integration constants.

The relations (A.29) between y_1, $p_{1,1}$ and w_1 are consistent with the small P expansion results of section 5 (see (5.14), (5.22), (5.30) and (5.38)):

$$
\begin{aligned}
y &= -\frac{1}{4}\log(4u) - \frac{1}{16}\mathrm{y}_1 u + \cdots\,, \qquad K(u) = -\frac{P^2}{2}\log u + 2P\mathrm{f}_{1,0}u + \cdots\,,\\
w &= \frac{1}{6}\mathrm{y}_1 u + \cdots\,, \qquad \Phi = -\frac{5}{4}\mathrm{y}_1 u\log u + \cdots\,. \qquad \text{(A.31)}
\end{aligned}
$$

In order to find perfect agreement, we must set

$$
K_* = \frac{P^2}{2}\log(8a) + O(P^4)\,, \qquad \mathrm{f}_{1,0} = aP\,, \qquad \mathrm{y}_1 = -\frac{8aP^2}{5K_*}\,. \qquad \text{(A.32)}
$$

Given these values and the relation $\mathrm{z}_{2,0} = P\mathrm{f}_{1,0} - \frac{5}{16}P^2\mathrm{y}_1$, one may also check that the expansion (A.25) is consistent with (5.29). To sum up, this comparison with the small P expansion has allowed us to fix two of the integration constants to $O(P^2)$, and presumably still more information could be extracted.

There are some differences between the small u expansion and the expressions found from perturbing in P. For example, from (5.14) one would expect $\mathrm{f}_{1,1} = 0$ while from (A.30) it is clearly not. However, $\mathrm{f}_{1,1}$ is $O(P^3)$ and hence would not be expected to show up at the order P^2 calculated in section 5.

Thus far, we have not considered the constraint eq. (3.7). We have calculated the expansion to one more order in u than was needed to extract useful information from the constraint. The constraint gives one relation between a, $\mathrm{f}_{1,0}$, y_1, y_3, and $\mathrm{f}_{3,0}$ which is complicated and is not reproduced here. The important point is that we have enough freedom to choose $\mathrm{f}_{1,0}$ and y_1 as in (A.32) and still satisfy the constraint.

The boundary conditions found above are the starting point for a numerical analysis which is to demonstrate that the $u \to 0$ and $u \to \infty$ asymptotics can be smoothly connected.

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